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LETTER TO THE EDITOR

Clifford periodicity from finite groups

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Abstract. We deduce the periodicity eight for the type of *Pin* and *Spin* representations of the orthogonal groups $O(n)$ from simple combinatorial properties of the finite Clifford groups generated by the gamma matrices. We also include the case of arbitrary signature $O(p, q)$. The changes in the type of representation can be seen as a rotation in the complex plane. The essential result is that adding a (+) dimension performs a rotation by $\pi/4$ in the counter-clockwise sense, but for each $(-)$ sign in the metric, the rotation is clockwise.

1. Introduction

The periodicity of Clifford algebras, first described by Atiyah *et al* [1],

$$C_{n+8} = C_n \otimes C_8$$

is a fundamental mathematical discovery. It is related to Bott's periodicity of the homotopy groups of the classical groups. It is essential in K-theory, in the solution by Adams of the vector field problem in spheres, etc. It is our aim to give a short proof of this important but simple property.

In this letter we shall obtain this periodicity from elementary properties of the representations of finite groups, namely the multiplicative groups generated by the 'Dirac gamma matrices' of the Clifford algebras. This finite group has been considered in the past [2], but not to our knowledge, applied to this problem.

If we have a positive quadratic form over the reals with isometry group $O(n)$, recall that the Clifford algebra is obtained by linearizing it, à la Dirac:

$$\sum (x^\mu)^2 = \sum (x^\mu \gamma_\mu)^2.$$

The algebra is generated by the γ_μ, where

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \tag{1}$$

There is a *finite* multiplicative group

$$\Gamma = \{\pm \mathbb{1}, \pm\gamma_\mu, \pm\gamma_\mu\gamma_\nu, \ldots, \pm\gamma_1, \ldots, \gamma_n \equiv \pm\gamma_{n+1}\} \tag{2}$$

with $\gamma_{n+1} = \Pi_n \gamma_\nu$, which generates the whole *Pin(n)* group, and the even part

$$\Gamma_0 = \{\pm \mathbb{1}, \pm\gamma_\mu\gamma_\nu, \pm\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho, \ldots\}$$

which generates *Spin(n)*.

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For our purposes we shall need two well known results from the representation theory of a finite group G (see e.g. Bacry [3] or Bröcker and Dieck [4]).

(A) Burnside theorem. *The group algebra is a direct sum of complete matrix algebras, and there are as many of these as there are classes of conjugate elements in G. Let $|G|$ be the order of G,*

$$|G| = \sum_{\text{classes}} (d_i)^2 \tag{3}$$

where i runs through the irreducible representations (irreps) of G, the same as the number of classes, and d_i is the dimension of the ith irrep.

Both Γ and Γ_0 are nearly Abelian in the sense that the commutator subgroup is very small. Hence the Abelianized quotient group is very large. Most of the irreps, in fact all except one or two, are therefore one-dimensional.

(B) The type, i, of a particular representation D. *This is given by the expression*

$$i(D) = \frac{1}{|G|}\sum_g \chi(g^2)$$
$$= \begin{cases} +1 & \textit{for real irreps} \\ 0 & \textit{for complex irreps} \\ -1 & \textit{for quaternionic, q-real, or quasireal irreps} \end{cases} \tag{4}$$

where $\chi_D(g) = \mathrm{Tr}(D(g))$ is the character of element g in the representation D. The result (4) also holds for compact groups [4].

In our case the sums in (4) are easy to compute because the anticommutativity (1) implies the squares $(\pm\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\ldots)^2 = \pm\mathbb{1}$, and the number of them is a simple combinatorial number.

2. Periodicity for $Pin(n)$

2.1. Even dimension

First, let $n = 2\nu$ be even. We have $|\Gamma| = 2^{n+1} = 2^{2\nu+1}$. The commutator $[\Gamma, \Gamma]$ is clearly $= \mathbb{Z}_2$. Hence the Abelianized quotient is half as large as Γ:

$$|\Gamma/[\Gamma,\Gamma]| = 2^{2\nu}.$$

There are also $2^{2\nu} + 1$ conjugation classes (all binomial terms plus one, which is the only nontrivial central), so there is a *unique* solution to Burnside's numerical equation (3)

$$2^{2\nu+1} = 2^{2\nu}\cdot 1^2 + 1\cdot(2^\nu)^2$$

and there is a single irrep of dimension 2^ν, the *(s)pin* representation, Δ.

The *type* of Δ is easy to compute; it is

$$i(\Delta) = \frac{2\cdot 2^\nu}{2^{2\nu+1}}\left[1 + \binom{2\nu}{1} - \binom{2\nu}{2} - \binom{2\nu}{3} + \binom{2\nu}{4} + \cdots\right].$$

The factor of two in the numerator is the $\pm$ sign in (2) and 2^ν is the dimension of Δ. It is clear why the signs alternate in blocks of two: $\gamma_\nu^2 = +1$ implies $(\gamma_\mu\gamma_\nu)^2 = -1$ which implies $(\gamma_\mu\gamma_\nu\gamma_\rho)^2 = -1$, etc. Hence,

$$i(\Delta) = \frac{1}{2^\nu}\left[1 - \binom{2\nu}{2} + \binom{2\nu}{4} - + \cdots\right] + \frac{1}{2^\nu}\left[\binom{2\nu}{1} - \binom{2\nu}{3} + - \cdots\right]$$
$$= \frac{1}{2^\nu}[\mathrm{Re}\,(1+\mathrm{i})^{2\nu} + \mathrm{Im}\,(1+\mathrm{i})^{2\nu}] = (\mathrm{Re}\, + \mathrm{Im}\,)[(1+\mathrm{i})/\sqrt{2}]^{2\nu}$$

which gives

$$i(\Delta) = (\mathrm{Re} + \mathrm{Im})\mathrm{e}^{2\pi \mathrm{i} n/8} \tag{5}$$

for $n = 2\nu$, even. So that for n even the periodicity is clearly seen to be eight,

$$i(\Delta) = \cos(2\pi n/8) + \sin(2\pi n/8).$$

Note that there are *no* complex irreps for the *Pin*(2ν) groups.

2.2. *Odd dimension*

The computation for $n = 2\nu + 1$ odd is similar:

$$|\Gamma| = 2 \cdot 2^{2\nu+1} = 2^{2\nu+2}.$$

The Burnside relation gives

$$2^{2\nu+2} = 2^{2\nu+1} \cdot 1^2 + 2 \cdot (2^\nu)^2.$$

There are now *two Pin*$(2\nu + 1)$ irreps of the same type; call them still Δ:

$$\begin{aligned} i(\Delta) &= \frac{2 \cdot 2^\nu}{2^{2\nu+2}} \left[1 + \binom{2\nu+1}{1} - \binom{2\nu+1}{2} - \binom{2\nu+1}{3} + \cdots \right] \\ &= \frac{1}{\sqrt{2}} (\mathrm{Re} + \mathrm{Im})[(1+\mathrm{i})/\sqrt{2}]^{2\nu+1} \end{aligned} \tag{6}$$

so that

$$i(\Delta) = \frac{1}{\sqrt{2}} (\cos(2\pi n/8) + \sin(2\pi n/8)) \tag{7}$$

for $n = 2\nu + 1$ odd. This, together with (5), completes the periodicity eight:

$$\begin{aligned} i(\Delta) &= 1, 1, 1, 0, -1, -1, -1, 0, 1, 1, 1, \ldots \\ n &= 0, 1, 2, 3, \quad 4, \quad 5, \quad 6, 7, 8, 9, 10, \ldots. \end{aligned}$$

The essential, simple result is that adding a dimension (in this 'Euclidean' case) corresponds to a rotation of $\pi/4$.

3. Periodicity for $Spin(n)$

Now we use the restricted finite Clifford group

$$\begin{aligned} &\Gamma_0 = \{\pm \mathbb{1}, \pm\gamma_\mu\gamma_\nu, \pm\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho, \ldots\} \\ &|\Gamma_0| = 2^n. \end{aligned}$$

Let $n = 2\nu$ even. The Burnside relation gives

$$2^{2\nu} = 2^{2\nu-1} \cdot 1^2 + 2 \cdot (2^{\nu-1})^2.$$

The two spin irreps are the traditional $\Delta^\pm$. So for n even their types are given by

$$\begin{aligned} i(\Delta^\pm) &= \frac{2 \cdot 2^{\nu-1}}{2^{2\nu}} \left[1 - \binom{2\nu}{2} + \binom{2\nu}{4} - \cdots \right] \\ &= \cos(2\pi n/8). \end{aligned} \tag{8}$$

For $n = 2\nu + 1$ odd, the Burnside relation gives

$$2^{2\nu+1} = 2^{2\nu} \cdot 1^2 + 1 \cdot (2^\nu)^2.$$

The type of the representation is then

$$i(\Delta^+) = \sqrt{2}\cos(2\pi n/8) \tag{9}$$

for n odd. Combining (8) and (9) we recover the usual *Spin(n)* periodicity eight:

$$\begin{aligned} i(\Delta^\pm) &= 1, 1, 0, -1, -1, -1, 0, 1, 1, 1, \ 0, \ldots \\ n \quad &= 0, 1, 2, \quad 3, \quad 4, \quad 5, 6, 7, 8, 9, 10, \ldots. \end{aligned}$$

The relation between *Pin*$(n-1)$ with *Spin(n)* is to be expected since the corresponding complete Clifford algebras coincide [5].

4. The case of signature

It is easy to extend the results above for a metric with signature (p, q) where p, q are *arbitrary* positive integers. Now we have more groups:

$$O(p, q) \qquad SO(p, q) \qquad SO_0(p, q)$$

where the $SO_0(p, q)$ is the connected part. Now the finite group Γ generates *Pin(p,q)*, but the restricted group, Γ_0, generates only *Spin(p,q)*, which covers $SO_0(p, q)$ twice.

The signature complication is inessential, as *each negative sign dimension corresponds to a* $\pi/4$ *rotation in the opposite (clockwise) sense*. To prove this, it is enough to reckon the type for the negative-definite metric, $(0, n)$. Now $(\gamma_\mu)^2 = -1$, so sets of odd numbers of γ change sign, but the even sets do not. Hence,

$$\text{Type}(0, n) = \mathcal{P}\left(\frac{(1-\mathrm{i})}{\sqrt{2}}\right)^n = \mathcal{P}\exp(-2\pi \mathrm{i} n/8) \tag{10}$$

where $\mathcal{P}$ is the projection (with the appropriate factor of $\sqrt{2}$ as before), (Re + Im) for the complete *Pin* group, and Re only for the *Spin* part.

As the angles add independently, we have

$$\begin{aligned} \text{Type}(p, q) &= \mathcal{P}[\exp(2\pi \mathrm{i} p/8)\exp(-2\pi \mathrm{i} q/8)] \\ &= \mathcal{P}[\exp(2\pi \mathrm{i}(p-q)/8)] \end{aligned} \tag{11}$$

which of course can be proved directly from the sums

$$\begin{aligned} &\left(1 + \binom{p-q}{1} - \binom{p-q}{2} - \binom{p-q}{3} + \binom{p-q}{4}\cdots\right) \\ &\left(1 + \binom{p}{1} - \binom{p}{2} - \binom{p}{3}\cdots\right)\left(1 - \binom{q}{1} - \binom{q}{2} + \binom{q}{3}\cdots\right). \end{aligned}$$

We have that *Pin*$(p, q) \neq$ *Pin(q,p)*, but the *Spin* groups are the same. The double covering of the connected part is unique, but the extensions from $O(p, q)$ and $O(q, p)$ are different.

Formula (11) is our final result. It shows eight-periodicity in the signature $(p-q)$, which is well known. We recall some consequences.

- The so-called split forms (p, p) and $(p+1, p)$ are real.
- The Lorentzian metric $(p, 1)$ has type $(p-1)$, so it is two in Minkowski space, regardless of whether it is (3, 1) or the light cone (2, 0).
- The same for the conformal extension $O(p, q) \to O(p+1, q+1)$, the type is still that of (p, q).
- The Lorentz groups $O(25, 1)$ and $O(9, 1)$ used in string theory are of the real type. This, no doubt, is crucial for the scale anomaly cancellation.
- The anomaly-free gauge group $O(32)$ used in type I and Heterotic string theory is also of the real type.

5. Final remarks

Clifford periodicity eight for the real orthogonal groups is an important phenomenon; so we find it satisfying to be able to provide a proof that is intrinsic, i.e., does not depend on the particular representation of the gamma matrices. It also covers the *Pin* as well as the *Spin* groups, and deals with the case of arbitrary signature.

The existence of two groups for the full orthogonal group has found an interesting application in paper [6]. In fact, the reflection properties of spinors *do* depend on the sign of the metric, and even in the 'skeleton' finite Clifford group this difference shows up.

We might mention another periodicity shown by one of us [7], which should be related to the case discussed here; namely the optical theorem in quantum mechanical scattering. This also depends on the dimension of the space with periodicity eight (although it has other factors, such as the volume of the sphere and the inverse of the momentum to some power, that depend on the dimension of the space as well). The formula reads [7]

$$\sigma_{tot} + 2\left(\frac{2\pi}{k}\right)^{(n-1)/2} \mathrm{Re}\,\{\mathrm{e}^{2\pi \mathrm{i}(n-1)/8} f(0)\} = 0 \qquad (12)$$

where σ_{tot} is the total elastic scattering cross section in n-dimensional space and $f(0)$ is the forward scattering amplitude. The similarity with the results above is striking and the reason, we think, is the same: the wavefunction is a kind of 'square root' of an orthogonal observable, and hence behaves like a spinor. This argument was already advanced in [7].

Finally, we call attention to the book [8] in which there is also a 'clock' with $\mathbb{Z}/8$ rotations.

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